

# Full Groups and Orbit Equivalence in Cantor Dynamics

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## Abstract

In this note we consider dynamical systems  $(X, G)$  on a Cantor set  $X$  satisfying some mild technical conditions. The considered class includes, in particular, minimal and transitive aperiodic systems. We prove that two such systems  $(X_1, G_1)$  and  $(X_2, G_2)$  are orbit equivalent if and only if their full groups are isomorphic as abstract groups. This result is a topological version of the well-known Dye's theorem established originally for ergodic measure-preserving actions.

## 1 Introduction

Denote by  $\text{Homeo}(X)$  the group of all homeomorphisms of a Cantor set  $X$ . Then the pair  $(X, G)$ , where  $G$  is a subgroup of  $\text{Homeo}(X)$ , is called a *Cantor dynamical system*. We would like to emphasize from the very beginning that, in contrast to the ergodic theory, the group  $G$  does not have to be countable.

For a point  $x \in X$ , denote by  $\text{Orb}_G(x) = \{g(x) | g \in G\}$  its  $G$ -orbit. Set also

$$[G] = \{\gamma \in \text{Homeo}(X) : \gamma(x) \in \text{Orb}_G(x) \text{ for all } x \in X\}.$$

Then  $[G]$  is a subgroup of  $\text{Homeo}(X)$ , which is called the *full group of  $G$* .

Two dynamical systems  $(X, G)$  and  $(Y, H)$  on the Cantor sets  $X$  and  $Y$  are called *orbit equivalent* if there is a homeomorphism  $\Lambda : X \rightarrow Y$  such that  $\Lambda(\text{Orb}_G(x)) = \text{Orb}_H(\Lambda(x))$  for all  $x \in X$ .

The notion of orbit equivalence and full groups appeared first in the context of ergodic theory in the seminal papers [D1] and [D2]. In these works Dye established a number of remarkable results that classified measure-preserving dynamical systems up to orbit equivalence. In particular, he showed the following result. Let two countable groups  $G$  and  $H$  act on a standard measure space  $(Y, \mu)$  by measure-preserving automorphisms. Then the full groups  $[G]$  and  $[H]$  (defined by measure-preserving transformations) are isomorphic as abstract groups if and only if the systems  $(Y, \mu, G)$  and  $(Y, \mu, H)$  are orbit equivalent. Furthermore, the algebraic isomorphism between full groups is always spatially generated by a map that implements the orbit equivalence.

This result implies that the full group “remembers” all the dynamical information which does not depend on the order of points within orbits. To the best of our knowledge there is still no a complete algebraic description of full groups. However, some partial results clarifying how different dynamical properties affect the algebraic structure of the full group have been earlier established, see, for example, [E1], [KT], and [Me].

We should point out that Dye’s theorem is a universal result as it holds in completely different dynamical setups. For example, there is a Borel version of Dye’s theorem [MilRos] established for full groups of Borel equivalence relations. The thesis of Miller [Mil] contains algebraic characterizations (in terms of full groups) of certain properties of underlying Borel dynamical systems.

In the context of Cantor dynamics, the topological version of Dye’s theorem was earlier obtained for minimal actions of locally finite groups and the group  $\mathbb{Z}$ , see [GPS]. In view of the work [GMPS], this result is expandable to minimal  $\mathbb{Z}^n$ -actions. Some algebraic properties of the full group  $[\mathbb{Z}]$  are present in the papers [M2] and [BM]. We should also mention the work [M1], where a version of Dye’s theorem is established for minimal actions of the group  $\mathbb{Z}$  on locally compact zero-dimensional spaces.

The main result of the present paper (Theorem 2.5) is the extension of Dye’s theorem on almost arbitrary Cantor systems. Namely, we prove that two Cantor systems  $(X, G)$  and  $(Y, H)$ , which meet some mild technical conditions, are orbit equivalent if and only if their full groups are isomorphic. This result shows that hyperfinite and non-hyperfinite actions can be already distinguished at the level of full groups (cf. [KT] for the ergodic case).

After the paper was submitted, we became aware of the work [R1] devoted to the reconstruction of Boolean algebras from their transformation groups. In particular, Theorem 4.5(c) in there implies the main result of the present paper (after the corresponding interpretation of the result). It can be derived from [R1, Theorem 4.5(c)] that if two Cantor systems have orbits at least of length three and the set of points with orbits of length six is nowhere dense, then any isomorphism between full groups is spatially generated. We would like to mention, though, that our proof is significantly different and shorter from that of [R1, Theorem 4.5(c)] and requires less prerequisites.

## 2 Spatial Realization

In this section we establish the main result of the paper. In our proof we will use a result of Fremlin [Fr, Theorem 384D] that states that algebraic isomorphisms between groups of automorphisms of complete Boolean algebras are always generated by an automorphism of the underlying algebras. We will apply this result to the case of full groups and show that the automorphism gives rise to a homeomorphism of the Cantor sets, which, in its turn, implements an orbit equivalence. The method of [Fr, Theorem 384D] has been already used in Cantor dynamics (see [BM]) to show that the commutator of the topological full group of minimal  $\mathbb{Z}$ -action is a complete invariant for flip conjugacy. It should be

noted that the Boolean algebra automorphism obtained in [BM] automatically turned out to be a homeomorphism (due to the definition of the topological full group). In the general case, we have to find an algebraic criterion for a set to be clopen. To achieve our program, we will need some notions of the theory of Boolean algebras.

Let  $X$  be a Cantor set. Recall that an open set  $A$  is called *regular open* if  $A = \text{int}(\overline{A})$ . Denote the family of all regular open sets by  $RO(X)$ . Notice that the family of clopen sets (denoted by  $CO(X)$ ) is contained in  $RO(X)$ .

Let  $\mathcal{A}$  be a Boolean algebra and  $\mathcal{H} \subset \mathcal{A}$ . Define  $\text{sup}(\mathcal{H})$  to be the smallest element of  $\mathcal{A}$  that contains all elements of  $\mathcal{H}$ . If  $\text{sup}(\mathcal{H})$  exists for any family  $\mathcal{H} \subseteq \mathcal{A}$ , then the Boolean algebra  $\mathcal{A}$  is called *complete*.

**Proposition 2.1.**  *$RO(X)$  is a complete Boolean algebra with Boolean operations given by*

$$A + B = \text{int}(\overline{A \cup B}), \quad A \cdot B = A \cap B, \quad A - B = A \setminus \overline{B}$$

*and with suprema given by  $\text{sup}(\mathcal{H}) = \text{int}(\overline{\bigcup \mathcal{H}})$ .*

*Proof.* See Theorem 314P of [Fr]. □

*Remark 2.2.* Notice that finite set-theoretical unions, intersections, and complements of clopen sets coincide with the corresponding Boolean operations.

For a homeomorphism  $\gamma$ , define its *support* as the set

$$\text{spr}(\gamma) = \text{int}(\overline{\{x \in X \mid \gamma(x) \neq x\}}).$$

Note that  $\text{spr}(\gamma)$  is a regular open set. Since every homeomorphism  $\gamma$  defines a Boolean algebra  $RO(X)$  isomorphism, we can also define the support of  $\gamma$  as the least regular open set  $S(\gamma)$  such that  $\gamma(V) = V$  for every regular open set  $V \subset X - S(\gamma)$  (see [Fr, Def. 381B]). It is not hard to check that  $S(\gamma) = \text{spr}(\gamma)$ .

We will also consider a point-wise support of the homeomorphism  $\gamma$  defined by  $\text{supp}(\gamma) = \{x \in X \mid \gamma(x) \neq x\}$ . We note that both sets  $\text{supp}(\gamma)$  and  $\text{spr}(\gamma)$  are  $\gamma$ -invariant and open. Furthermore,  $\text{supp}(\gamma) \subset \text{spr}(\gamma)$  and for any point  $x \in \text{supp}(\gamma)$  there is a clopen neighborhood  $W$  such that  $\gamma(W) \cap W = \emptyset$ .

**Definition 2.3.** Following [Fr, Def. 382O], we say that a group  $\Gamma \subset \text{Homeo}(X)$  *has many involutions* if for any regular open set  $A$  there is an involution  $\pi \in \Gamma$  with  $\text{spr}(\pi) \subset A$ .

**Proposition 2.4.** *Let  $(X, G)$  be a Cantor dynamical system.*

(1) *If for every clopen set  $A$  there is a point  $x \in A$  whose  $G$ -orbit intersects  $A$  at least twice, then the full group  $[G]$  has many involutions.*

(2) *If every orbit of  $G$  has the length at least three, then the supports of involutions (with clopen supports) from  $[G]$  generate the Boolean algebra  $CO(X)$  (with the standard set-theoretical operations).*

*Proof.* (1) For a clopen set  $A$ , find a point  $x \in A$  and  $g \in G$  such that  $g(x) \in A$  and  $x \neq g(x)$ . Choose a clopen set  $V \ni x$  with  $V, g(V) \subset A$  and

$V \cap g(V) = \emptyset$ . Define  $\pi|V = g|V$ ,  $\pi|g(V) = g^{-1}|g(V)$ , and  $\pi = id$  elsewhere. Then  $\pi$  is an involution supported by  $A$ .

(2) Fix a clopen set  $A$  and a point  $x \in A$ . If there is an element  $g_x \in G$  with  $g_x(x) \in A$  and  $g_x(x) \neq x$ , then choose a clopen set  $U_x$  such that  $g_x(U_x) \cap U_x = \emptyset$  and  $U_x, g_x(U_x) \subset A$ . If for all  $g \in G \setminus \{id\}$ ,  $g(x) \notin A$ , then we choose two elements  $g_x^{(1)}, g_x^{(2)} \in G$  with  $\{x, g_x^{(1)}(x), g_x^{(2)}(x)\}$  being distinct points. Choose a set  $U_x \subset A$  so that  $\{U_x, g_x^{(1)}(U_x), g_x^{(2)}(U_x)\}$  are mutually disjoint and  $g_x^{(1)}(U_x), g_x^{(2)}(U_x)$  are subsets of  $X \setminus A$ . Take a finite subcover  $\{U_{x_1}, \dots, U_{x_n}\}$  of  $A$ . If  $g_{x_i}(U_{x_i}) \subset A$ , then we may construct an involution  $\pi_i$  as in (1) with  $x \in spr(\pi)$  and  $spr(\pi)$  being a clopen subset of  $A$ . If the trajectory of  $x$  intersects  $A$  only once, then by using the elements  $g_{x_i}^{(1)}$  and  $g_{x_i}^{(2)}$ , we may construct two involutions  $\pi_i^{(1)}$  and  $\pi_i^{(2)}$  as in (1) so that  $spr(\pi_i^{(1)})$  and  $spr(\pi_i^{(2)})$  are clopen sets and  $spr(\pi_i^{(1)}) \cap spr(\pi_i^{(2)}) = U_{x_i}$ . This implies that the set  $A$  is represented as a union and intersection of a finite number of clopen supports of involutions.  $\square$

As a corollary, we get that every transitive system  $(X, G)$  with infinite orbits has many involutions. Furthermore, the clopen supports of the involutions from  $[G]$  generate  $CO(X)$ .

**Theorem 2.5.** *Let  $(X_1, G_1)$  and  $(X_2, G_2)$  be Cantor dynamical systems such that each  $G_i$ -orbit contains at least three points and the full group  $[G_i]$  has many involutions for  $i = 1, 2$ . Then  $(X_1, G_1)$  and  $(X_2, G_2)$  are orbit equivalent if and only if  $[G_1]$  and  $[G_2]$  are isomorphic as abstract groups.*

*Furthermore, for every isomorphism  $\alpha : [G_1] \rightarrow [G_2]$  there is a homeomorphism  $\Lambda : X_1 \rightarrow X_2$  such that  $\alpha(g) = \Lambda g \Lambda^{-1}$  for all  $g \in [G_1]$ .*

*Proof.* It is obvious that the orbit equivalence implies the isomorphism of full groups. Conversely, let  $\alpha : [G_1] \rightarrow [G_2]$  be a group isomorphism. Since both groups  $[G_1]$  and  $[G_2]$  have many involutions, there is an automorphism of Boolean algebras  $\Lambda : RO(X_1) \rightarrow RO(X_2)$  such that  $\alpha(g)(V) = \Lambda g \Lambda^{-1}(V)$  for any  $g \in [G_1]$  and any  $V \in RO(X_2)$  (see Theorem 384D in [Fr]). Our goal is to show that  $\Lambda$  gives rise to a homeomorphism of the Cantor sets by establishing that  $\Lambda(CO(X_1)) = CO(X_2)$ .

Let  $G$  stand for either of the groups  $G_1$  and  $G_2$ . We will establish some general properties of the full group  $[G]$ . For any regular open set  $V$ , set  $\Gamma_V = \{\gamma \in [G] : spr(\gamma) \subset V\}$ . It will be clear from the context which group  $G_1$  or  $G_2$  is meant. The following lemma immediately follows from the proof of Theorem 384D of [Fr] (see items (g) and (i) therein). We also refer the reader to the proof of Theorem 5.8 in [BM] to see how this result can be obtained from the scratch in the case of topological full groups of minimal  $\mathbb{Z}$ -systems.

**Lemma 2.6.** (1) *Let  $V \in RO(X_1)$ . Then  $\Lambda(V) = \sup\{spr(\alpha(\pi)) : \pi \in [G_1] \text{ is an involution with } spr(\pi) \subset V\}$ .*

(2) *If  $\pi \in [G_1]$  is an involution, then  $spr(\alpha(\pi)) = \Lambda(spr(\pi))$  and*

$$\alpha(\Gamma_{spr(\pi)}) = \Gamma_{spr(\alpha(\pi))}.$$

**Definition 2.7.** For an involution  $\pi \in [G]$ , set  $W_\pi = \Gamma_{spr(\pi)}$ . Then, in view of (2) in Lemma 2.6,  $\alpha(W_\pi) = W_{\alpha(\pi)}$ . We note that the proof of [Fr, Theorem 384D] contains a precise algebraic description of the subgroups  $W_\pi$ . See also Corollary 2.10 in [R2].

Our goal now is to give an algebraic criterion for  $spr(\pi)$  to be a clopen set. We will need the following two lemmas. For any group  $\Gamma$  in  $[G]$  denote by  $\Gamma^\perp$  the commutator of  $\Gamma$  in  $[G]$ , i.e.  $\Gamma^\perp = \{\rho \in [G] : \rho\gamma = \gamma\rho \text{ for any } \gamma \in \Gamma\}$ . For every set  $V \in RO(X)$ , set also  $V^\perp = X - V$ .

**Lemma 2.8.** *If  $V$  is a regular open set, then*

$$\Gamma_V^\perp = \{\gamma \in [G] : spr(\gamma) \subset X - V\} = \Gamma_{V^\perp}.$$

*Proof.* Suppose that  $\gamma \in [G]$  such that  $spr(\gamma) \subset X - V$ . Fix any element  $\rho \in \Gamma_V$ . Then  $spr(\rho) \subset V$ . Hence  $spr(\rho) \cap spr(\gamma) = \emptyset$ . This implies that  $\rho$  and  $\gamma$  commute.

Conversely, if  $\gamma \in \Gamma_V^\perp$  and  $spr(\gamma) \not\subset X - V = X \setminus \overline{V}$ , then  $spr(\gamma) \cap \overline{V} \neq \emptyset$ . Since  $spr(\gamma)$  is an open set, we get that  $spr(\gamma) \cap V \neq \emptyset$ . Since the pointwise support  $supp(\gamma)$  of  $\gamma$  is an open dense subset of  $spr(\gamma)$ , we get that  $supp(\gamma) \cap V \neq \emptyset$ .

Take any clopen subset  $W$  of  $supp(\gamma) \cap V$  such that  $\gamma(W) \cap W = \emptyset$ . Using the fact that the group  $[G]$  has many involutions, find an involution  $\rho$  supported by  $W$ . Clearly,  $\rho \in \Gamma_V$ . Take any clopen set  $O \subset W$  with  $\rho(O) \cap O = \emptyset$ . Then  $\rho\gamma(O) \cap \gamma\rho(O) = \emptyset$ , which is a contradiction.  $\square$

For a set  $F \subset [G]$ , denote by  $\langle F \rangle$  the subgroup generated by the elements of  $F$ .

**Lemma 2.9.** *A regular open set  $V$  is clopen if and only if for any involution  $\pi \notin \langle \Gamma_V, \Gamma_{V^\perp} \rangle$  there is an element  $\rho \in \langle \Gamma_V, \Gamma_{V^\perp} \rangle$  such that  $h = \pi\rho$  is an involution and*

(i) *if  $g \in \Gamma_V \cap W_h$ , then  $h^{-1}gh \in \Gamma_{V^\perp}$ ;*

(ii) *if  $g \in \Gamma_{V^\perp} \cap W_h$ , then  $h^{-1}gh \in \Gamma_V$ .*

*Proof.* (1) First of all assume that  $V$  is clopen. Set  $R = \langle \Gamma_V, \Gamma_{V^\perp} \rangle$ . Let  $\pi \notin R$ . Set

$$A = V \cap \pi(V) \cap spr(\pi) \text{ and } B = V \cap \pi(X \setminus V) \cap spr(\pi).$$

Since the set  $V$  is clopen, both of the sets  $A$  and  $B$  are open. Furthermore,  $B \neq \emptyset$  as  $\pi \notin R$  and  $\pi(A) = A$ . Observe that  $\overline{A} \cap \overline{B} = \emptyset$ . Indeed, if otherwise, take  $x \in \overline{A} \cap \overline{B}$ , then there are two sequences  $\{a_n\} \subset A$  and  $\{b_n\} \subset B$  with  $a_n \rightarrow x$  and  $b_n \rightarrow x$ . However, by the definition of  $A$  and  $B$ , we get that  $\{\pi(a_n)\} \subset V$  and  $\{\pi(b_n)\} \subset X \setminus V$ . As the set  $V$  is clopen, we get that  $\pi$  is not continuous at  $x$ , which is a contradiction.

Thus, we can choose a clopen set  $O \subset V$  such that  $A \subset O$  and  $O \cap \overline{B} = \emptyset$ . Note that  $\pi(O) = O$ . Set  $\rho_1|_O = \pi^{-1}|_O$  and  $\rho_1 = id$  elsewhere. Then  $\pi\rho_1|(V \setminus B) = id$ . Clearly,  $\pi\rho_1$  is an involution and  $\rho_1 \in \Gamma_V$ .

Repeating the same arguments with the set  $X \setminus V$  and the involution  $\pi\rho_1$ , we find an element  $\rho_2 \in \Gamma_{V^\perp}$  such that  $\pi\rho_1\rho_2|(V^\perp \setminus \pi(B)) = id$ . Set  $\rho = \rho_1\rho_2 \in R$  and  $h = \pi\rho$ . Observe that  $spr(h) = B \cup \pi(B)$  and  $h|_B = \pi|_B$ .

Now if  $g \in W_h \cap \Gamma_V$ , then  $spr(g) \subset B$ . Hence  $spr(hgh^{-1}) \subset \pi(B)$  and  $hgh^{-1} \in \Gamma_{V^\perp}$ . The condition (ii) can be established in a similar way.

(2) Conversely, assume that  $V$  is a non-closed regular open set. Set

$$B = X \setminus (V \cup (X - V)).$$

Then  $B$  is a non-empty closed set. Note also that  $B = \overline{V} \cap \overline{X - V}$ .

(2-i) Fix a point  $x \in B$ . If there is an element  $g \in [G]$  with  $g(x) \in V$  or  $g(x) \in X - V$ . Take a small clopen neighborhood  $U$  of  $x$  with  $g(U) \cap U = \emptyset$ . Define an involution  $\pi \notin R$  by setting  $\pi|_U = g|_U$ ,  $\pi|_{g(U)} = g^{-1}|_{g(U)}$ , and  $\pi = id$  elsewhere.

Without loss of generality, we will assume that  $g(x) \in V$ . Fix any element  $\rho \in R$ . Since  $\rho(x) = x$ , we get that  $\pi\rho(x) \neq x$ . Set  $h = \pi\rho$ . Hence,  $h(O) \cap O = \emptyset$  for some clopen neighborhood of  $x$ . Note that  $O \cap V \neq \emptyset$ . Take any involution  $q \in \Gamma_V$  with  $spr(q) \subset O$ . It follows that  $q \in W_h$  and  $spr(hqh^{-1}) \subset h(O) \subset V$ . Hence  $hgh^{-1} \in \Gamma_V$ , which contradicts the condition (i) of the lemma. The case when  $g(x) \in X - V$  is proved similarly.

(2-ii). Now consider the situation when  $g(x) \in B$  for any point  $x \in B$  and any element  $g \in [G]$ . This means that  $g(B) = B$  for all  $g \in G$ . Since, every  $G$ -orbit has the length at least three, choose three different points  $\{x_1, x_2, x_3\} \subset B$  from the same  $G$ -orbit. By the standard arguments, find two involutions  $\pi_1$  and  $\pi_2$  from  $[G]$  with clopen supports such that  $\pi_1(x_1) = x_2$  and  $\pi_2(x_2) = x_3$ .

We claim that there is a point  $x \in \{x_1, x_2, x_3\}$  and a homeomorphism  $\pi \in \{\pi_1, \pi_2, \pi_1\pi_2\}$  such that for every clopen neighborhood  $U$  of  $x$  either  $\pi(U \cap V) \cap V \neq \emptyset$  or  $\pi(U \cap V^\perp) \cap V^\perp \neq \emptyset$ .

Assume the converse. Since  $\pi_1(B) = B$ , there is a clopen neighborhood  $U_1$  of  $x_1$  with  $\pi_1(U_1 \cap V) \subset V^\perp$  and  $\pi_1(U_1 \cap V^\perp) \subset V$ . In the same way, there is a clopen neighborhood  $U_2$  of  $x_2$  with  $\pi_2(U_2 \cap V) \subset V^\perp$  and  $\pi_2(U_2 \cap V^\perp) \subset V$ . It follows that  $\pi_2\pi_1(U \cap V) \subset V$  for some clopen subset  $U \ni x$  of  $U_1$ , which is a contradiction.

Without loss of generality, we will assume that  $\pi(U \cap V) \cap V \neq \emptyset$  for every clopen neighborhood of  $x$ . This means that every open neighborhood of  $x$  contains an element  $g \in \Gamma_V$  with  $\pi g \pi^{-1} \in \Gamma_V$ . Repeating arguments from (2-i), we get that for any  $\rho \in R$  there is  $g \in \Gamma_V$  with  $(\pi\rho)g(\pi\rho)^{-1} \in \Gamma_V$ .

This proves the necessity of the condition (i). The necessity of the condition (ii) can be established in a similar way.  $\square$

*Continuation of the proof.* It follows from the lemmas above that if a clopen set  $V$  is the support of an involution  $\pi$ , then  $\Lambda(V) = spr(\alpha(\pi))$  is a clopen set as well.

Since  $CO(X_i)$  is generated by the clopen supports of involutions and  $\Lambda$  is a Boolean algebra isomorphism, we conclude that  $\Lambda(CO(X_1)) = CO(X_2)$ . This implies that  $\Lambda$  defines a homeomorphism of  $X_1$  and  $X_2$ .

If  $y = g(x)$  for some  $x \in X_1$  and  $g \in [G_1]$ , then  $\Lambda(y) = \Lambda g(x) = \alpha(g)\Lambda(x)$ . Thus,  $\Lambda(x)$  and  $\Lambda(y)$  belong to the same  $G_2$ -orbit. Hence,  $\Lambda$  implements the orbit equivalence between  $(X_1, G_1)$  and  $(X_2, G_2)$ .  $\square$

**Definition 2.10.** Let  $(X, G)$  be a Cantor dynamical system. Then the *topological full group* of  $G$  (in symbols  $[[G]]$ ) is formed by all elements  $\gamma \in [G]$  for which there is a clopen partition  $\{U_1, \dots, U_n\}$  of  $X$  and elements  $g_1, \dots, g_n \in G$  such that  $\gamma|_{U_i} = g_i|_{U_i}$  for every  $i$ .

*Remark 2.11.* Note that if  $[G]$  has many involutions, then so does the topological full group  $[[G]]$  (see [GPS] for the definition). So we can follow the lines of the proof of Theorem 2.5 to show that any group isomorphism between topological full groups is spatially generated.

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